- Half of calculus is based on differential $\frac{\partial F}{\partial x_i}$
- Taking differential of a map $F: M \to N$ needs new concept — tangent vectors.
- All these can be formulated in terms of more advanced language.

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① vector bundle

<u>Def</u> A real vector bundle of rank k over mfd M is a smooth map $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ s.t. $\exists$ an open cover $\{U_\alpha\}_\alpha$ of M and a collection of smooth bijective

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \simeq U_\alpha \times \mathbb{R}^k \quad \text{(local trivilization)}$$

satisfying:

(i) $\Phi_\alpha\big|_{\pi^{-1}(\{x\})} : \pi^{-1}(\{x\}) \longrightarrow \{x\} \times \mathbb{R}^k$ ← fiberwise

(ii) $\forall\ \alpha, \beta$, $\quad \Phi_\beta \circ \Phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^k$

has the form

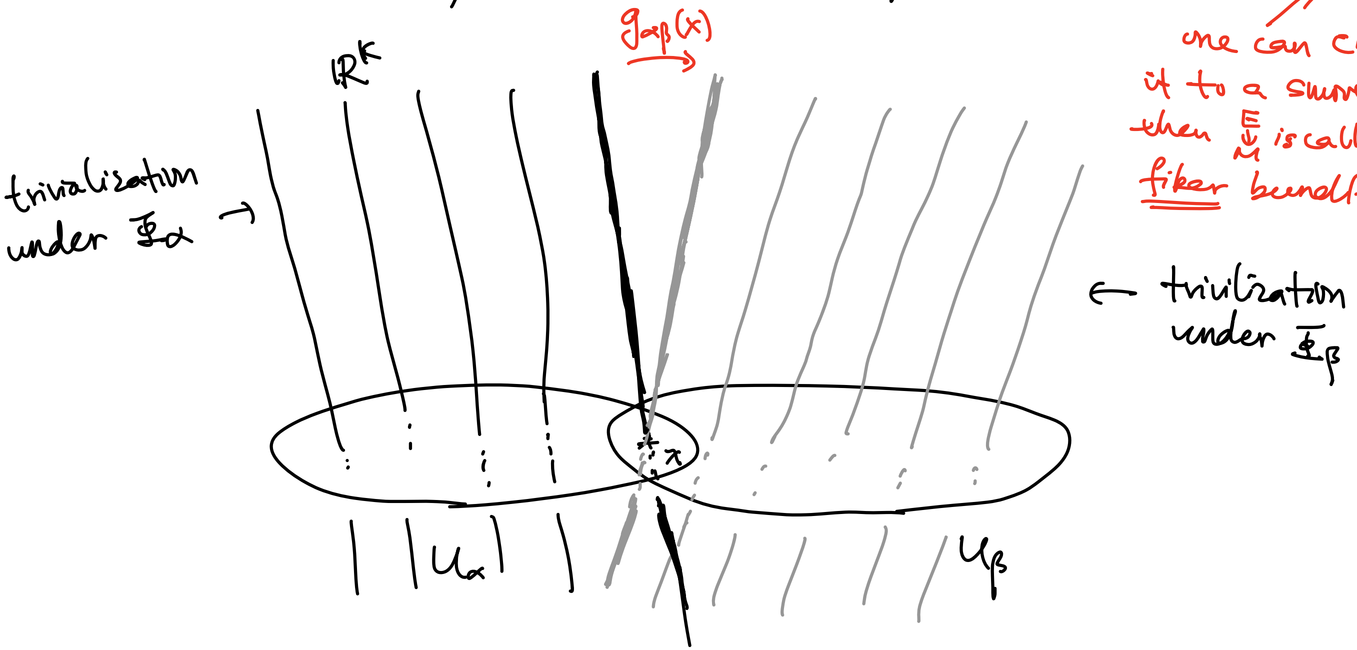
b/c (i)

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x, v) = (x, \underline{g_{\alpha\beta}}(x) \cdot v)$$

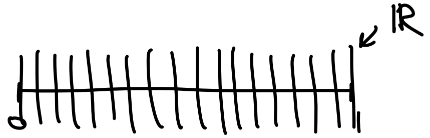
transition map $\in GL(n, k)$.

where for every $x \in U_\alpha \cap U_\beta$, $g_{\alpha\beta}(x) : \mathbb{R}^k \to \mathbb{R}^k$ is a <u>linear</u> map

one can change it to a smooth map then $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ is called a <u>fiber</u> bundle.

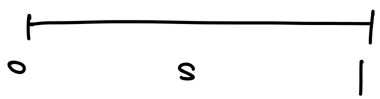


eg Consider quotient manifold $E = \dfrac{[0,1]\times\mathbb{R}}{(0,t)\sim(1,-t)}$

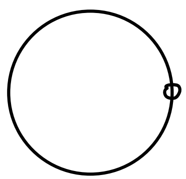
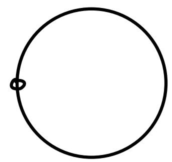


$[0,1]\times\mathbb{R}(t)$

$$\begin{tikzcd}
{[0,1]\times\mathbb{R}} \arrow[r, "q", "\text{quotient}"'] \arrow[d, "\pi_1"] & E \arrow[d, dashed, "\pi"] \\
{[0,1]} \arrow[r, "e^{2\pi i s}"] & S^1 = \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$



- Define $\pi([(s,t)]) := e^{2\pi i s}$
- Consider the following open cover of S^1:

$U_1 =$  $= S^1 \setminus \{+1\}$ $\qquad$ $U_2 =$  $= S^1 \setminus \{-1\}$